Self-organized criticality in linear interface depinning and sandpile models

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The dynamics of an elastic interface profile $h(x,t)$ under a driving force increasing at rate c, a restored force $-\epsilon h$, and disorder is investigated. Using perturbation theory and functional renormalization group the phase diagram and the scaling exponents, up to the first order in $\varepsilon = 4-d$, are obtained. The model is found to be critical in the double limit $\epsilon \to 0$ and $c/\epsilon \to 0$ and belongs to a different universality class as that of constant force models. It is shown that undirected sandpile models with stochastic rules and linear interface models with extremal dynamics belong to this new universality class.

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The problem of interface roughening in the present of quenched disorder is a topic of recent interest, due to its importance as a paradigm in condensed matter physics and due to the broad range of applications. In general a d-dimensional self-affine interface, described by a single-valued function $h(x,t)$, evolves in a $(d+1)$-dimensional medium. Usually some form of disorder affects the motion of the interface leading to its roughening. Earlier studies [1] focuses on time-independent uncorrelated disorder but most recent studies analyses the motion of interfaces under quenched disorder [2–4]. In the present of quenched disorder an constant force driving two universality classes has been found [5]. One is described by the Kardar-Parisi-Zhang equation [1] with quenched noise. In this case the interface is pinned by paths on a directed percolation cluster of pinning sites [6]. The second class is described by the Edwards-Wilkinson (EW) equation [7] with quenched noise, linear interface depinning (LID).

The interface may be driven either by extremal dynamics [8–10] or by constant force [11–13]. While constant force models has been extensively studied in the literature, either by functional renormalization group (FRG) [14,15] or numerical simulations [14], extremal dynamics models are less known. However, in the last years extremal dynamics models have gain more attention due to its relation with the theory of self-organized criticality (SOC).

SOC was introduced to explain the critical behavior of a vast class of driven dissipative system which evolve into a critical state [16]. In its early state it was believed that such a critical state is insensitive to changes in control parameters and no fine-tuning is needed. More recent interpretations of this phenomena have shown that criticality in SOC systems is obtained after some control parameters, for instance the driving and dissipation rates, are fine-tuned to zero [17].

In the present work we focus our attention in the motion of a d-dimensional interface profile obeying the following equation,

$$\lambda\partial_t h = \Gamma\nabla^2 h + F - \epsilon h + \eta(x,h). \qquad (1)$$

with a force increasing at constant rate, $F = ct$. Here λ is a friction coefficient, Γ is the surface tension, c is the driving rate, and ϵ a nonnegative constant. The random force $\eta(x,h)$ is Gaussian distributed with zero mean and

$$\langle \eta(x,h)\eta(x',h')\rangle = \delta^d(x-x')\Delta(h-h'), \qquad (2)$$

where $\Delta(h)$ is a monotonically decreasing function.

The aim of this work is to show that models described by this equation are in a different universality class as constant force models, and that undirected sandpile models with stochastic rules and LID models with extremal dynamics belongs to this new universality class. The motivation for the analogy between sandpile models and LID models is based in a work by Paczusky and Boettcher [18], where it is shown that a one dimensional critical slope sandpile model is in the same universality class as the depinning transition of a $d+1$ interface dragged at one end. Although this work pointed out the analogy between LID and sandpile models its analysis was limited to one dimension. We onclude that the existence of "spontaneous" criticality in LID with extremal dynamics, as in sandpile model [17], is just a consequence of the unprecise definition of these models.

If the force F is constant and $\epsilon = 0$ then eq. (1) is reduced to the EW equation with quenched noise. This case has been extensively studied in the literature [14,15]. A depinning transition takes place at certain critical field force F_c determined by the disorder. For $F < F_c$ the interface is pinned after certain finite time while for $F > F_c$ it moves with finite average velocity which scales as $v \sim (F - F_c)^\beta$.

When $\epsilon > 0$ and the force increases at rate c then the interface is never pinned by disorder, but always moves with a finite average velocity v. A perturbative solution of eq. (1) can thus be found expanding $h(x,t)$ around the flat co-moving interface vt. Taking $h(x,t) = vt + y(x,t)$ we obtain the following equation for $y(x,t)$

$$\lambda\partial_t y = \Gamma\nabla^2 y + (c-\epsilon v)t - \epsilon h - \lambda v + \eta(x, vt+h). \qquad (3)$$

The average velocity is obtained using the constraint $\langle y(x,t)\rangle = 0$. For this purpose is better to work with the equation for the Fourier transform $\tilde{h}(k,\omega)$. The effective external field $(c-\epsilon v)t$ gives a singular term of the order of ω^{-2}. This singular term predominates in the low frequency limit resulting, after imposing $\langle \tilde{y}(k,\omega)\rangle = 0$,

$$v = \frac{c}{\epsilon}, \tag{4}$$

which is valid to all orders of perturbation expansion.

Another exact result can be obtained if one computes the low-frequency and long-wavelength susceptibility. Adding a source term $\varphi(x,t)$ to the right hand side of eq. (3) and going to the Fourier space one obtains the generalized response function

$$\tilde{G}(k,\omega) = \left\langle \left.\frac{\tilde{h}(k,\omega)}{\tilde{\varphi}(k,\omega)}\right|_{\tilde{\varphi}=0} \right\rangle = \frac{1}{[\tilde{G}_0(k,\omega)]^{-1} - \tilde{\Sigma}(k,\omega)}, \tag{5}$$

where

$$[\tilde{G}_0(k,\omega)]^{-1} = \Gamma k^2 - i\lambda\omega + \epsilon \tag{6}$$

is the bare correlator and $\tilde{\Sigma}(k,\omega)$ is the "self-energy". Since $\tilde{\Sigma}(0,0) = 0$ and $\tilde{G}_0(0,0)^{-1} = \epsilon$ it results that the low-frequency and long-wavelength susceptibility (or simply the susceptibility) is given by

$$\chi = \tilde{G}(0,0) = \epsilon^{-\gamma}, \quad \gamma = 1. \tag{7}$$

This result is also exact to all orders of perturbation expansion. Thus, when $\epsilon \to 0$ the susceptivility diverges and, therefore, the system is critical.

To go further we perform a FRG analysis of the problem, following the general ideas developed for the constant force case [15]. We construct the generating functional $Z = \int DhD\hat{h}\exp(S)$ with action

$$S = \int d^dxdti\hat{h}[\lambda\partial_t h - \Gamma\nabla^2 h - F + \epsilon h - \eta(x,h)], \tag{8}$$

where $\hat{h}(x,t)$ is an auxiliary field. After averaging over disorder an expansion around the mean-field (MF) solution yields a generating functional with the low frequency form [15] $\bar{Z}\int DyD\hat{y}\exp(\bar{S})$, where the effective action $\bar{S}$ is given by

$$\begin{aligned}\bar{S} = &-\int d^dxdt\{[F - F_{\mathrm{MF}}(v)]\hat{y}(x,t) + \hat{y}(x,t)(\lambda\partial_t - \Gamma\nabla^2 + \epsilon)y(x,t)\} \\ &+\frac{1}{2}\int d^dxdt_1dt_2\hat{y}(x,t_1)y(x,t_2)C[v(t_1-t_2) + y(x,t_1) - y(x,t_2)],\end{aligned} \tag{9}$$

where y and $\hat{y}$ are coarse grained versions of $h - vt$ and $-i\hat{h}$, respectively, and $C(h)$ is the MF correlation function. Two differences appear with the constant force case. First, in our case $F = ct$ and $F_{\mathrm{MF}}(v)$, the MF force corresponding to a velocity v, is given by $F_{\mathrm{MF}}(v) = c_{\mathrm{MF}}(v)t$. Moreover, we have obtained that $c = v\epsilon$ (see eq. (4)) exactly, within and beyond the MF approximation. Hence $F - F_{\mathrm{MF}}(v) = 0$. Second, in the Gaussian part of the effective action there is an extra term associated with the restored force, characterized by the coefficient ϵ. As it was shown above the susceptibility diverges when $\epsilon \to 0$. Thus, ϵ is the control parameter of the interface described by eq. (1). On the contrary in the constant force case $\epsilon = 0$ and $F - F_{\mathrm{MF}}(v)$ is the control parameter.

The RG transformations are carried out as follows. We integrate out the degrees of freedom in a momentum shell near the cutoff Λ and rescale $x \to bx$, $t \to b^z t$, $y \to b^\zeta y$, and $\hat{y} \to b^{\theta-d}\hat{y}$, where $b = \mathrm{e}^l$ with $l \to 0$. As usual the cutoff Λ appears because we start our analysis from a coarse-grained equation, where we cannot resolve spatial details smaller than Λ^{-1}.

The renormalization of the ϵ term yields

$$\frac{d\epsilon}{dl} = (\theta + z + \zeta)\epsilon, \tag{10}$$

which implies that the correlation length scale as $\xi \sim \epsilon^{-\nu}$ with $\nu = 1/(\theta + z + \zeta)$. Since $\theta + z + \zeta) = 2$ [15] we finally obtain $\nu = 1/2$, which differs from the one obtained in the constant force case, due to the existence of different control parameters. On the contrary other scaling exponents results identical. For instance [15]

$$\zeta = \frac{\varepsilon}{3}, \quad z = 2 - \frac{2}{9}\varepsilon. \tag{11}$$

On the other hand, vt and y must scale in the same, so that $v \to b^{\zeta - z}v$ yielding

$$\frac{dv}{dl} = (\zeta - z)v. \tag{12}$$

Thus, to reach the critical state both ϵ and v should be fine-tunned to zero, i.e. criticality is obtained in the double limit $\epsilon \to 0$ and $v = c/\epsilon \to 0$. From eq. (12) we define the characteristic velocity $v_c \sim \epsilon^\beta$ with

$$\beta = \nu(z - \zeta). \tag{13}$$

Note that in constant force LID the average interface velocity in the supercritical regime is given by $v \sim$

$(F - F_c)^\beta$. On the contrary, in the present model the average interface velocity is fixed through eq. (4) and the exponent β just characterizes the ϵ dependence of the characteristic velocity v_c, which delimit the subcritical and supercritical regimes. For $v \gg v_c$ the noise term in eq. (3)is approximately annealed obtaining the EW equation with annealed noise. In this case $\zeta = 0$ for $d \geq 2$. The driving field predominates over disorder and, therefore, the model is supercritical.

In the subcritical regime $\epsilon > 0$ and $v \ll v_c$ the dynamics takes place in the form of avalanches, characterized by the avalanche size distribution $P(s) = s^{-\tau} g(s/s_c)$, where $s_c \sim \epsilon^{-1/\sigma}$ is a cutoff avalanche size. In the critical state $s_c \sim L^D$, where D is the avalanche dimension, and $\xi \sim L$ leading to the scaling relation $\sigma = 1/D\nu$. Another scaling relation is obtained taking into account that $\chi = \langle s \rangle$, leading to $\gamma = (2 - \tau)/\sigma$. On the other hand, for $d < d_c$, the avalanche dimension and the roughness exponent are related via $D = d + \zeta$ [10]. Using these scaling relations and the values for γ, ν, ζ, and z computed above we obtain

$$D = d + \zeta, \qquad \tau = 2(1 - D^{-1}). \tag{14}$$

To investigate the analogy with sandpile models let us analyze a discretized variant of eq. (1). In the cellular automaton version of eq. (1) one defines the total force

$$F_i = \sum_{nn} H_j - 2dH_i + ht - \epsilon H_i + \eta_i(H_i), \tag{15}$$

and sites where $F_i > 0$ are updated in parallel, advancing the interface by one $H_i \rightarrow H_i + 1$. Here $\sum_{nn} H_j - 2dH_i$ is a discretized Laplacian, where the sum runs over nearest neighbors. Instead of follow the evolution of the interface profile H_i one may keep track of the total force F_i. The evolution rules for the total force F_i are given by: 1- on each step F_i is increased by h in all sites and 2- all sites where $F_i > 0$ are updated in parallel according to the toppling rule

$$\begin{array}{l} F_i \rightarrow F_i - 2d - \epsilon + \eta_{ai} \\ F_{nn} \rightarrow F_{nn} + 1, \end{array} \tag{16}$$

where $\eta_{ai} = \eta_i(H_i + 1) - \eta_i(H_i)$ is a zero mean uncorrelated annealed noise and nn denotes nearest neighbors.

From a simple inspection of eq. (16) one can see that the total force follows the evolution rules of an undirected sandpile automaton with an annealed noise, under a driving field h and with dissipation rate per toppling event ϵ. This class of sandpile models has been studied by Vespignani and Zappery [17], using mean-field and field theories. They have obtained that $\gamma = 1$ and $\nu = 1/2$ are exact in all dimensions. Moreover, their field theory predicts an upper critical dimension $d_c = 4$. These results are in agreement with those obtained here and strongly suggest that both models are in the same universality class.

In the mean-field and field theory by Vespignani and Zappery conservation (understanding conservation as the balance between input and output energy) is a necessary condition for stationarity, which implies that the density of toppling (active) sites is given by $\rho_a = h/\epsilon$. The equivalent condition in our approach is found in eq. (4), which is a consequence of the balance between the driving force ht and the average restored force ϵvt. The balance of these two forces is thus our necessary condition for stationarity. Moreover, since the interface advances only at active sites (those where $F_i > 0$), in one unit, then $v = \rho_a$, making the connection between our approach and that of Vespignani and Zapperi. Is also important to note that $v \leq 1$, which is consistent with the fact $\rho_a \leq 1$.

Now, we proceed to show that the extremal LID model corresponds to the critical state of eq. (1), i.e. $h, \epsilon \rightarrow 0$ and $v \rightarrow 0$. The condition $h \rightarrow 0$ carry as a consequence that, if at time step t_0 there are no active sites ($F_i < 0$ for all sites) then at time step $t_1 = -F_j(t_0)/h$ the site j, with the maximum total force, will be active. In the language of interface depinning the interface at site j will advance one unit, in the sandpile language the site j will topple. It is thus clear that if no active site is present the system will follows extremal dynamics. Now, what happened during the evolution of an avalanche?

The active site j will transfer energy to its nearest neighbors, which at the same time may become active, and so on, an avalanche is generated. It is thus possible that at certain time step t there will more that one active site. These sites will be updated in parallel according to the evolution rules described above. However, the order in which these sites are updated is not important. The process of toppling can never transform any active site, different from itself, in inactive and, therefore, the others active sites will remain active. On the other hand, the energy transferred to its neighbors is constant, independent of the total force at this site. Moreover, the site with the maximum total force will be always among the set of active sites. Hence, one can arrange the sequence of toppling events, of actives sites at time step t, in such a way that always the site with the maximum total force topples.

Finally, in the double limit $h \rightarrow 0$, $\epsilon \rightarrow 0$ the total force is reduced to the local force

$$F_i \rightarrow f_i = \sum_{nn} H_j - 2dH_i + \eta_i(H_i), \tag{17}$$

Hence, in the critical state of the cellular automaton version of eq. (1) one may arrange the sequence of toppling events in such a way that the site which topples has the maximum local force f_i, which is the extremal dynamics variant of LID. We thus conclude that extremal dynamics models corresponds to the critical state of the LID model described by eq. (1).

In table I some numerical estimates for the LID model with extremal dynamics and the Manna d-state model,

the prototype of stochastic sandpile model, are given. Our FRG estimates from eqs. (11) and (14) are also shown for comparison. Increasing the lattice dimension the FRG predictions get closer to the numerical estimates, obtaining a complete agreement in three dimensions. Using the numerical estimates for τ and D, reported for the LID model with extremal dynamics, we have tested the scaling relation $(2-\tau)D = \gamma\nu^{-1} = 2$, within the numerical error it is fulfilled. In all cases the difference in the numerical estimates for both models are contained in the error bars, which implies that they are in the same universality class. Our predictions are thus confirmed with the numerical simulations.

In the discussion we have not included the Bak-Tang-Wiesenfeld (BTW) model neither a sandpile model with stochastic dissipation introduced by Chessa *et al* [19]. In the first case because the BTW model is deterministic and it is not clear yet if stochastic and deterministic models are in the same universality class. The LID model introduced here was mapped into a sandpile model with annealed noise and, therefore, our conclusions cannot be extended to deterministic models. In the second case because according to the simulations by Chessa *et al* [19] the upper critical dimension for their stochastic model is 6, in disagreement with $d_c = 4$ as obtained from our FRG analysis and the numerical simulations of the d-state Manna model [20]. On the theoretical side, $d_c = 4$ is in agreement with previous reports by Díaz-Guilera, using a dynamic renormalization group approach for the BTW and Zhang models [21], and by Vespignani *et al* [22].

In summary, we have shown that the existence of "spontaneous" criticality in LID with extremal dynamics, as in sandpile model [17], is just a consequence of the unprecise definition of these models. SOC corresponds to the onset of nonlocality in the dynamics of the interface. Nonlocality, and hence criticality, is obtained by fine tuning the control parameters, precisely as in continuous phase transitions. The extremal dynamics corresponds with a fine tuned interface depinning transition at constant velocity. It was also demonstrated that LID with extremal dynamics and undirected sandpile models with stochastic rules belong to the same universality class, which is different from that of constant force LID.

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d	Model	τ	z	D	$(2-\tau)D$	Ref.
1	LID	1.13(2)		2.23(3)	1.94(7)	[10]
	FRG	1	$\frac{4}{3} \approx 1.33$	2	2	
2	LID	1.29(2)		2.75(20)	1.95(7)	[10]
	Manna	1.273	1.500	2.750	2.00	[20]
	FRG	$\frac{5}{4} = 1.25$	$\frac{14}{9} \approx 1.56$	$\frac{8}{3} \approx 2.67$	2	
3	LID			$3.34(1)^*$		[14]
	Manna	1.40	1.75	3.33	2.00	[20]
	FRG	$\frac{7}{5} = 1.4$	$\frac{16}{9} \approx 1.78$	$\frac{10}{3} \approx 3.33$	2	

TABLE I. Scaling exponents for the LID model with extremal dynamics and the Manna d-state model. Results obtained here using FRG are shown for comparison. * It was computed using the scaling relation $D = d + \zeta$ and the reported value of ζ.